THE

COMPLETE SOLUTION

OF

NUMERICAL EQUATIONS:

IN WHICH,

BY ONE UNIFORM PROCESS,

THE IMAGINARY AS WELL AS THE REAL ROOTS
ARE EASILY DETERMINED.

BY

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THE object of the following researches is not so much to ascertain the number of the imaginary roots of any proposed equation, as to determine their numerical values to any extent.

Professor YOUNG, of Belfast, in his "*Theory and Solutions of Equations of the higher orders*," and in his "*Researches respecting the Imaginary Roots of Numerical Equations*," which forms an Appendix to the former valuable Work, has so ably and so fully discussed the recent discoveries and inquiries of BUDAN, FOURIER, and STURM, respecting the character and situation of the roots of equations, that my investigations have been confined almost entirely to the developement of a process for finding the numerical values of the imaginary roots.

When these investigations were commenced, I contemplated only the determination of the values of the imaginary roots of equations, but from the form of an imaginary root which I was led to employ, viz. $a + \sqrt{-\beta}$*, it was easy to see that the values of the *real* as well as those of the *imaginary roots* would be obtained by one and the same process,—the *criterion* of the character of the roots being the *sign* of the value of β. The *sign* of β will obviously indicate whether the roots are real or imaginary, and when the sign of β is negative, the *magnitude* of $-\beta$ will indicate whether these real roots are equal, unequal, or nearly equal.

If an equation has two roots nearly equal, they will be readily separated by the method I have adopted; for if the value of $-\beta$ is positive, and the first significant figure of its decimal value be preceded by $2n$ ciphers, the two roots will necessarily be identical as far as $n-1$ places of decimals.

The process employed in Note A for the simultaneous determination of *all the three* roots of a cubic equation is remarkable for its simplicity and elegance, and the convenient arrangement of the work, may eventually promote the general adoption of the method in our elementary treatises on Algebra.

It is almost unnecessary to remark, that if a simple and practical process could be devised for removing several of the terms of the higher equations, the determination of their imaginary roots would be greatly facilitated, more especially as Mr. Weddle's ingenious method of approximation might then be employed with much advantage.

* The form $a + \sqrt{\beta}$ might have been used instead of $a + \sqrt{-\beta}$.

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January, 1849.

NEW METHOD

OF

RESOLVING NUMERICAL EQUATIONS.

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1. THE investigation of a simple method for the complete solution of numerical equations of all degrees has occupied the attention of many eminent mathematicians. The beautiful Theorem of STURM for determining the number of real roots of a numerical equation of any degree whatever, is theoretically complete, though in practice it becomes very laborious when applied to equations of the fifth and higher degrees; while HORNER's elegant method of approximating to the values of the *real* roots of any numerical equation, leaves nothing to be desired, so far as regards the determination of these roots to any extent that may be proposed. Though much has been effected in this department of science by WARING, LAGRANGE, FOURIER, BUDAN, STURM, HORNER, ATKINSON, YOUNG, DAVIES, LOCKHART, and WEDDLE, the solution of numerical equations cannot be considered as fully accomplished without the knowledge of some easy method of determining the values of their *imaginary* roots, kindred to that of HORNER for approximating to their real roots.

2. In his "*Traité de la Résolution des Equations Numériques de tous les Degrés*," the celebrated LAGRANGE has given a method of finding the imaginary roots of equations, which is universally acknowledged to be almost impracticable in its application to equations even of the fourth and fifth degrees, and it has never been employed as an instrument of calculation, on account of the difficulty of finding the coefficients of the unknown quantity in the transformed equation. The principle of LAGRANGE's method is to transform any proposed equation into another whose roots are the *squares of the differences* of the roots of the given equation, but the coefficients of the several powers of the unknown quantity in the transformed equation have a most formidable appearance, especially those which result from the transformation of an equation of the fifth degree.—(See the *Philosophical Transactions* for 1763, or *Traité des Equations Numériques,* 1808, note III, p. 111, where these coefficients, first determined by WARING, are given.)

3. After giving the coefficients of the different powers of the unknown quantity in the transformed equations for determining the imaginary roots of equations of the third and fourth degrees, LAGRANGE himself remarks, (*ibid*, p. 43),

"*On pourrait de même trouver les conditions qui rendent les racines des équations du cinquième degré toutes réelles, ou en partié réelles et en partié imaginaires; mais, comme dans ce cas, l'équation des differences monterait au degré* $\frac{5 \cdot 4}{2} = 10$, *le calcul deviendrait extrêmement prolix et embarrassant.*"

4. In the following pages it is proposed to develope a new method of finding not only the values of the *real* roots of equations, but especially the values of the *real* and *imaginary* parts of the *imaginary roots* of equations of all degrees, which appears to possess considerable advantages over the method of LAGRANGE, and which is more simple and effective than any process that has hitherto been devised.

5. It has been already remarked that the principle of LAGRANGE's method is the transformation of any proposed equation into another whose roots are the squares of the differences of the roots of the given equation; therefore it is obvious that, if the proposed equation has two imaginary roots of the form $a + \beta \sqrt{-1}$ and $a - \beta \sqrt{-1}$, the transformed equation of differences will have one real root of the form $-4\beta^2$, since the difference of $a + \beta \sqrt{-1}$ and $a - \beta \sqrt{-1}$ is $\pm 2\beta \sqrt{-1}$, and its square is $-4\beta^2$. By this method, therefore, we must *first* find the value of β, or the imaginary part of the imaginary root, and having obtained the value of β, we are then directed to substitute $a + \beta \sqrt{-1}$ for x in the proposed equation, and to separate the resulting equation into two others,—the one having its terms all real, and the other having each of its terms multiplied by $\sqrt{-1}$. In this manner we get two equations in a of the form

$$a^m + \mathrm{P}a^{m-1} + \mathrm{Q}a^{m-2} + \mathrm{R}a^{m-3} + \ldots\ldots\ldots = 0,$$

$$ma^{m-1} + pa^{m-2} + qa^{m-3} + ra^{m-4} + \ldots\ldots\ldots = 0;$$

in which the coefficients P, Q, R, *etc.*, and p, q, r, *etc.*, are given in terms of the coefficients of the unknown quantity in the given equation, and of β. Now if we give to β, one of its values previously found, it will be obvious that these two equations, which exist simultaneously, will have a common measure. If then the greatest common measure of the polynomial expressions in the first members of these equations be found and equated to zero, we shall obtain an equation in a and β, from which, β being known, a may be found.

Such is the operose method proposed by LAGRANGE for the determination of the imaginary roots of equations of all degrees; but had that distinguished analyst viewed the subject in a slightly different aspect, he would have obtained very different and much less complicated results.

6. In the method we now propose to develope, we may find, at pleasure, either the real or the imaginary part of the imaginary roots of an equation, according as we eliminate β or a from two simultaneous equations which involve both these quantities. The elimination of β from these equations, will, *in all cases*, be much more readily effected than the elimination of a, and therefore the *real* part of the imaginary roots of equations, ought to be the *first* object of research, and *then* the imaginary part will be readily found from one or other of the specified equations, or from the greatest common measure of their first members when equated to zero. The principle of the method is founded on the following familiar proposition: *Any numerical equation whatever being given, to transform it into another whose roots shall be less or greater than the roots of the given equation, by a given quantity.*

7. Let the given equation be

$$x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \ldots\ldots + sx + t = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then the transformed equation whose roots are each less by aquantity α, than the roots of equation (1), will be

$$(x' + \alpha)^m + a(x' + \alpha)^{m-1} + b(x' + \alpha)^{m-2} + \ldots\ldots + s(x' + \alpha) + t = 0,$$

which, by expanding the several powers of $x + \alpha$, and arranging the result, becomes

$$x'^m + \mathrm{A}x'^{m-1} + \mathrm{B}x'^{m-2} + \mathrm{C}x'^{m-3} + \ldots\ldots + \mathrm{S}x' + \mathrm{T} = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

where $x' = x - \alpha$, and the several coefficients A, B, C, *etc.*, are functions of α, such that

$$\left.\begin{aligned}
\mathrm{A} &= m\alpha + a \\
\mathrm{B} &= \frac{m(m-1)}{1.2}\alpha^2 + (m-1)a\alpha + b \\
\mathrm{C} &= \frac{m(m-1)(m-2)}{1.2.3}\alpha^3 + \frac{(m-1)(m-2)}{1.2}a\alpha^2 + (m-2)b\alpha + c \\
\mathrm{D} &= \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}\alpha^4 + \frac{(m-1)(m-2)(m-3)}{1.2.3.}a\alpha^3 + \frac{(m-2)(m-3)}{1.2}b\alpha^2 \\
&\qquad + (m-3)c\alpha + d \\
&\vdots \\
\mathrm{S} &= m\alpha^{m-1} + (m-1)a\alpha^{m-2} + (m-2)b\alpha^{m-3} + \ldots\ldots 2r\alpha + s \\
\mathrm{T} &= \alpha^m + a\alpha^{m-1} + b\alpha^{m-2} + c\alpha^{m-3} + \ldots\ldots\ldots\ldots\ldots\ s\alpha + t.
\end{aligned}\right\} \ldots (3).$$

Let now $\alpha + \sqrt{-\beta}$, $\alpha - \sqrt{-\beta}$, r_1, r_2, *etc.*, denote the roots of equation (1), where the roots of the form $\alpha \pm \sqrt{-\beta}$* will be *real* or *imaginary*, according as the value of β is *negative* or *positive*, and the resulting formulas and equations will determine the *real* as well as the *imaginary* roots of any equation ; then will that equation be equal to the continued product of the binomial factors

$$x - (\alpha + \sqrt{-\beta}), \quad x - (\alpha - \sqrt{-\beta}), \quad x - r_1,\ x - r_2,\ etc.$$

The product of the first two of these factors is $x^2 - 2\alpha x + \alpha^2 + \beta$, and let the product of all the other factors be represented by the expression

$$x^{m-2} + a'x^{m-3} + b'x^{m-4} + c'x^{m-5} +, etc. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathcal{A}).$$

Multiply together these two expressions, equate their product to zero, and we get the equation

$$\left.\begin{aligned}
&x^m + (a' - 2\alpha)x^{m-1} + (b' - 2a'\alpha + \alpha^2 + \beta)x^{m-2} + \{c' - 2b'\alpha + a'(\alpha^2 + \beta)\}x^{m-3} + \ldots \\
&\quad + \{r' - 2q'\alpha + p'(\alpha^2 + \beta)\}x^2 + \{-2r'\alpha + q'(\alpha^2 + \beta)\}x + r'(\alpha^2 + \beta)
\end{aligned}\right\} = 0 \ldots\ldots\ldots (4).$$

This equation ought to be identical with (1), and by equating the coefficients of the same powers of x in (1) and (4), we get

$$\left.\begin{aligned}
a &= a' - 2\alpha \\
b &= b' - 2a'\alpha + \alpha^2 + \beta \\
c &= c' - 2b'\alpha + a'(\alpha^2 + \beta) \\
&\vdots \\
r &= r' - 2q'\alpha + p'(\alpha^2 + \beta) \\
s &= - 2r'\alpha + q'(\alpha^2 + \beta) \\
t &= + r'(\alpha + \beta)
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

* The usual form of an imaginary root is $\alpha + \beta\sqrt{-1}$, but the form we have adopted here is better suited to our purpose, and it may be reduced to the ordinary form, by extracting the square root of β when it is positive, and placing that root before the imaginary symbol $\sqrt{-1}$.

In equation (3) substitute for $a, b, c, etc,$, their values as obtained in (5): then we shall have the following remarkable relations among A, B, C, *etc.*, the coefficients of the several powers of the unknown quantity in the transformed equation (3), whose roots are less by α than the roots of the given equation (1), namely,

$$\left.\begin{array}{l} \mathrm{A} = f_1\alpha \\ \mathrm{B} = f_2\alpha + \beta \\ \mathrm{C} = f_3\alpha + \mathrm{A}\beta \\ \mathrm{D} = f_4\alpha + \mathrm{B}\beta - \beta^2 \\ \mathrm{E} = f_5\alpha + \mathrm{C}\beta - \mathrm{A}\beta^2 \\ \mathrm{F} = f_6\alpha + \mathrm{D}\beta - \mathrm{B}\beta^2 + \beta^3 \\ \mathrm{G} = f_7\alpha + \mathrm{E}\beta - \mathrm{C}\beta^2 + \mathrm{A}\beta^3 \\ etc. \qquad etc. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where $f_1\alpha = (m-2)\alpha + a'$

$$f_2\alpha = \frac{(m-2)(m-3)}{1.2}\alpha^2 + (m-3)a'\alpha + b'$$

$$f_3\alpha = \frac{(m-2)(m-3)(m-4)}{1.2.3}\alpha^3 + \frac{(m-3)(m-4)}{1.2}a'\alpha^2 + (m-4)b'\alpha + c'$$

$$f_4\alpha = \frac{(m-2)(m-3)(m-4)(m-5)}{1.2.3.4}\alpha^4 + \frac{(m-3)(m-4)(m-5)}{1.2.3}a'\alpha^3 + \frac{(m-4)(m-5)}{1.2}b'\alpha^2 + (m-5)c'\alpha + d'$$

$$f_5\alpha = \frac{(m-2)(m-3)(m-4)(m-5)(m-6)}{1.2.3.4.5}\alpha^5 + \frac{(m-3)(m-4)(m-5)(m-6)}{1.2.3.4}a'\alpha^4$$
$$+ \frac{(m-4)(m-5)(m-6)}{1.2.3}b'\alpha^3 + \frac{(m-5)(m-6)}{1.2}c'\alpha^2 + (m-6)d'\alpha + e';$$

which may be continued at pleasure.

8. From these remarkable relations of the coefficients A, B, C, *etc.*, which have resulted from the transformation of the proposed equation into another whose roots are each less than the roots of that equation by α, the real part of the imaginary root, we shall obtain the necessary equations for the determination of the imaginary roots. We must not omit to remark that the two equations which we shall obtain from the results of the preceding transformation of the given equation are precisely those mentioned by LAGRANGE, and which we have given in art. 5, p. 4. These equations may readily be obtained by substituting $\alpha + \sqrt{-\beta}$ or $\alpha - \sqrt{-\beta}$ for x in the proposed equation, and separating the real terms from those which are multiplied by $\sqrt{-1}$, and thus forming two equations involving α and β. The preceding formulas, however, will furnish these equations, in all cases, without any substitution, and we shall now proceed to apply them to the complete solution of some numerical equations of the third, fourth, and higher degrees.

I. CUBIC EQUATIONS.

9. Let the cubic equation be

$$x^3 + ax^2 + bx + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let r, $\alpha + \sqrt{-\beta}$ and $\alpha - \sqrt{-\beta}$ be its three roots, one of which (r) is necessarily real, and the two others will be real or imaginary, according as the sign of the value of β is $-$ or $+$. Then making $m = 3$ in equations (6), and recollecting that, in this case, all the coefficients of the several powers of x in the expression ($\mathcal{A}$) are zero, with the exception of a', we have $f_2\alpha = 0$ and $f_3\alpha = 0$; therefore we have

$$\mathrm{B} = \beta \quad \text{and } \mathrm{C} = \mathrm{A}\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From these two equations eliminate β, and we get the relation

$$AB - C = 0 \dots\dots (3).$$

In equations (3) art. 7, let $m = 3$, then we get

$$\begin{aligned} A &= 3\alpha + a \\ B &= 3\alpha^2 + 2a\alpha + b \\ C &= \alpha^3 + a\alpha^2 + b\alpha + c. \end{aligned}$$

Substituting these values of A, B, C in the relation (3), gives

$$\alpha^3 + a\alpha^2 + \frac{a^2 + b}{4}\alpha + \frac{ab - c}{8} = 0 \dots\dots (4),$$

an equation which will furnish the value of α, the rational part of the two other roots of the proposed equation.

If we substitute the values of A, B, C in equations (2) then

$$3\alpha^2 + 2a\alpha + b = \beta \dots\dots (5),$$

$$\alpha^3 + a\alpha^2 + b\alpha + c = \beta(3\alpha + a) \dots\dots (6),$$

and these are precisely the two equations that result from substituting either $\alpha + \sqrt{-\beta}$ or $\alpha - \sqrt{-\beta}$ for x in the given equation (1), and separating the rational from the irrational terms.

Now if the rational part of these binomial roots is to be *first* found, we must eliminate β from equations (5) and (6); but if β is the first object of research, then α must be eliminated from these equations. We have already eliminated β, and obtained equation (4) for the determination of α, and if we eliminate α, the resulting equation in β, will be found to be

$$\beta^3 + \frac{a^2 - 3b}{2}\beta^2 + \frac{(a^2 - 3b)^2}{16}\beta + \frac{4(a^2 - 3b)(b^2 - 3ac) - (9c - ab)^2}{192} = 0 \dots\dots (7).$$

This elimination is obviously not so readily effected as the elimination of β, and ought therefore to be avoided. If we change the signs of the alternate terms of (7), and then compare the transformed equation with Lagrange's equation in v, (see page 44), it will be seen that the roots of the equation in v are precisely *four* times the roots of equation (7) thus modified.

Having found the value of α from equation (4), the value of β will be found very readily from either (5) or (6). But to simplify the formula for obtaining the value of β, let us subtract eq. (4) from eq. (6); then the difference is

$$\frac{9c - ab}{8} - \frac{a^2 - 3b}{4}\alpha = \beta(3\alpha + a) \dots\dots (8).$$

This is equivalent to finding the greatest common measure of (5) and (6), and the value of β may be determined from the simplest of the three equations (5), (6), and (8).

If the proposed cubic is complete in all its terms, it will frequently be advantageous to divest it of its second term, for then we have $a = 0$, and the modified equations for finding α and β will be

$$\alpha^3 + \frac{b}{4}\alpha - \frac{c}{8} = 0 \dots\dots (9),$$

$$\beta^3 - \frac{3b}{2}\beta^2 + \frac{9b^2}{16}\beta - \frac{4b^3 + 27c^2}{64} = 0 \dots\dots (10);$$

or in terms of α,

$$\beta = 3\alpha^2 + b \dots\dots (11),$$

or

$$\beta = \frac{b}{4} + \frac{3c}{8\alpha} \dots\dots (12).$$

10. If we compare the equation (1), divested of its second term, viz.

$$x^3 + bx + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

with the equation (9), it will be seen that the roots of (13) are just *twice* the roots of (9), and since the alternate terms of these two equations have contrary signs, it is evident that if r_1, r_2, r_3 denote the roots of eq. (13); then will the roots of eq. (9) be $-\frac{r_1}{2}$, $-\frac{r_2}{2}$ and $-\frac{r_3}{2}$. Hence the values of a will be half those of x, with contrary signs.

11. We shall now apply these formulæ to one or two examples, and when the proposed equation is to be divested of its second term, the approximation to the root of the transformed equation, will be continued in a connected form, having the appearance of only one single operation.

EXAMPLE I.

Solve completely the cubic equation $x^3 - 6x - 6 = 0$.

Let r, $a + \sqrt{-\beta}$ and $a - \sqrt{-\beta}$ be the roots of the proposed equation, then since the last term is *negative*, the real root r will be *positive*, and the rational part a of the two other roots, whether they be real or imaginary, will be *negative* and equal to $-\frac{r}{2}$. The value of β will indicate whether the two other roots are real or imaginary, according as its sign is $-$ or $+$. When the second term is absent in any equation, the real root may first be found, as in the following operation.*

1 + 0	—6	— 6	(2·8473221019
2	4	— 4	
2	—2	— 10	
2	8	9152	
4	6	— 848	
2	544	714304	
6·8	1144	—133696	
8	608	127795423	
7 6	1752	—5900577	
8	3376	5495637	
8 44	178576	—404940	
4	3392	366430	
8 48	181968	—38510	
4	596 8 9	36643	
8 527	182564 8 9	—1867	
7	597 3 8	1832	
8 534	183162 2|7	—35	
7	25 6|2	18	
|85|41	183187 8|9	—17	
	25 6|2	16	
	183213|5	—1	
	1|7		
	183215|2		
	1|7		
	183217		

Hence the real root of the equation is 2·8473221019, and therefore the rational part of the two other roots is $a = -\frac{r}{2} = -1{\cdot}4236610509$. By equation (12), $\beta = \frac{b}{4} + \frac{3c}{8a}$; hence the following operation for finding β.

* The student is supposed to be acquainted with the usual methods of determining the initial figure of the root of an equation, as well as familiar with Horner's method of continuous approximation. Those who may find themselves deficient in these operations, may profitably consult a valuable treatise on the "Theory and Solution of Equations", by Professor YOUNG, of Belfast.

```
— 1·4,2,3,6,6,1,0,5,0,9 )   —2·2500000000  (    1·580432366
                             1 4236610509     —1·5         = ¼ b
                             ------------     -------------
                              8263389491      ·080432366 = β
                              7118305254
                              ----------
                              1145084237
                              1138928811
                              ----------
                                 6155426
                                 5694644
                                 -------
                                  460782
                                  427098
                                  ------
                                   33684
                                   28473
                                   -----
                                    5211
                                    4270
                                    ----
                                     940
                                     854
                                     ---
                                      86
                                      85
```

The positive value of β indicates that the two remaining roots are imaginary, and their values are therefore

$$-1{\cdot}4236610509 \pm \sqrt{-{\cdot}080432366} \text{ or } -1{\cdot}4236610509 \pm {\cdot}283606004\sqrt{-1}.$$

EXAMPLE II.

Solve completely the cubic equation $x^3 - 17x^2 + 54x - 350 = 0$.

Divest the equation of its second term, and then find the root of the transformed equation by a single operation in the following manner.

```
1 — 17        + 54              — 350   (5⅔    or     5·66666666666
     5·6̇      — 64 2̇           — 57·9̇25̇
  -----       ------            ---------            ( 9·28740194295
— 11·3̇       — 10·2̇           — 407·925             ---------------
     5·6̇      — 32·1̇              348                 14·95406860961 = real root.
  -----       ------            ---------------
—    5·6̇      — 42·3̇           — 59·925925925̇        2)—9·28740194295
     5·6̇         81                41·221333333       ---------------
  -----         -----            ---------------         — 4·64370097147
     0           38 6̇          —18·704592592              5·66666666666
     9          162                17·104085333          --------------
  -----        ------            ---------------          1·02296569519 = a,
     9         200·66̇          —1 600507259
     9           5 44               1·513517570
  -----        ------             ------------
    18         206·106̇              —86989689
     9           5 48                 86569167
  -----        ------                --------
   27 2        211·5866̇              —420522
      2          2 2144                 216434
  -----        --------               -------
   27 4        213·80106̇             —204088
      2          2 2208                194791
  -----        ---------              -------
   27 68       216 02186 6̇              —9297
       8            19492 9               8657
  ------       -------------            -----
   27 76       216·21679 56̇              —640
       8            19497 8                433
  ------       -------------             -----
   27 847      216·41177|3                —207
        7            1114|4                195
  -------      -------------             -----
   27 854      216·42291|7                 —12
        7            1114|4                 11
  -------      -------------
   2|78|61     2 1,6,4,3,4 0,6
```

the rational part of the two other roots. The value of β will be found from eq. (12) as below.

tr

$$c = -407\cdot\dot{9}2\dot{5}$$

		$-407\cdot925$
		3
	8)	$-1223\cdot777$
$-434\cdot670097147$)		$-152\cdot972222222$ ($32\cdot941876138$
		139 311029144 $-10\cdot583333333 = \frac{1}{4}b$
		13 661193078 $22\cdot358542805 = \beta$
		9 287401943
		4 373791135
		4 179330874
		194460261
		185748039
		8712222
		4643701
		4068521
		3714960
		353561
		325059
		28502
		27862
		640
		464
		176
		139
		37

The positive value of β indicates that the two remaining roots are imaginary, and the three roots of the proposed equation are therefore

$$14\cdot95406860961 \text{ and } 1\cdot02296569519 \pm \sqrt{-22\cdot358542805}.$$

EXAMPLE III.

Solve completely the equation $x^3 - 7x + 7 = 0$.

One real root of this equation is negative, since the last term is positive; and we shall approximate to the negative root without changing the signs of the alternate terms.

1 + 0	−7	+ 7·000000000 (− 3·0489173395 = r
− 3	9	− 6
− 3	2	1·000000
− 3	18	− 814464
− 6	200000	185536000
− 3	3616	− 166382592
− 904	203616	19153408
− 4	3632	−18791228
− 908	207248	362180
− 4	73024	−208875
− 9128	20797824	153305
− 8	73088	−146213
− 9136	2087091 2	7092
− 8	823 0	−6266
−91,44	2087914 2	826
	823	−627
	208873 7	199
	9	−188
	208874 6	11
	9	
	2,0,8,8,7,5	

Hence $a = -\frac{r}{2} = 1\cdot5244586698$, and the value of β is found below from the equation

$$\beta = \frac{b}{4} + \frac{3c}{8a}.$$

```
1·5244586698 ) 2·6250000000 ( 1·7219227074
               1 5244586698
               11005413302
               10671210689
                 334202613
                 304891734
                  29310879
                  15244587
                  14066292
                  13720128
                    346164
                    304892
                     41272
                     30489
                     10783
                     10671
                       112
                       106
                         6
```

$$\begin{array}{rl} & 1{\cdot}7219227074 \\ - & 1{\cdot}75 \qquad = \tfrac{1}{4}b \\ \hline - & {\cdot}0280772926 = \beta. \end{array}$$

Hence the roots are all real, and two of them have the same initial figure, since $\sqrt{-\beta} = {\cdot}167562802$; therefore

$a + \sqrt{-\beta} = 1.524458669 + {\cdot}167562802 = 1{\cdot}692021471$

$a - \sqrt{-\beta} = 1{\cdot}524458669 - {\cdot}167562802 = 1{\cdot}356895867,$

and the negative root has been found above $= -3{\cdot}048917339$.

The separation of the nearly equal roots of equations is completely effected by this method of solution, and we shall apply it to an additional example having two roots nearly equal to each other.

EXAMPLE IV.

Find all the roots of the equation $x^3 + 11x^2 - 102x + 181 = 0$.

The real root of this equation is negative, since the sign of the last term is positive; hence changing the signs of the alternate terms we have the subjoined operation.

```
1 − 11        − 102          − 181 ( 3⅔  or        3·66666666666
     3·6̇        − 26·8̇        − 472·59̇2̇
   − 7·3̇        − 128·8̇        − 653·592592592  ( 13·77598229514
     3·6̇        − 13·4̇            346·666666666    17·44264896180
   − 3·6̇        − 142.3̇        − 306·925925925
     3·6̇          169              274·719666666   2 ) 13·77598229514
     0             26·6̇          − 32·206259259       6·88799114757
    13            338                29·653299666       3·66666666666
    13            364·6̇6         − 2·552959592        3·22132448091 = a,
    13             27·79              2·133559708
    26            392·456̇          − 419399884      the rational part of the two
    13             28·28              384260160      remaining roots.
   397            420·7366̇          − 35139724
     7              2·8819              34159698
   404            423·61856̇          − 980026
     7              2·8868               853999
  4117            426·50536 6̇        − 126027
     7              ·20657 5              85400
  4124            426·71194 1̇6        − 40627
     7              ·20660 0               38430
 41315            426·91854|1           − 2197
     5                3719|2               2135
 41320            426·95573|3            − 62
     5                3719|2                 43
 4|13|25          426·9929|3                 19
                       33|0
                  426·9962|3
                       33|
                  4,2,6,9,9,9,5
```

$c = 653 \cdot 5925925925$
3

$8\ )\ 1960 \cdot 7777777777$

$6 \cdot 88799114757\ )$	$245 \cdot 0972222222$	$(\ 35 \cdot 58326614703$
	$206 \cdot 6397344271$	$-\ 35 \cdot 58333333333 = \frac{1}{4} b$
	$38 \cdot 4574877951$	$-\ \cdot 00006718630 = \beta.$
	$34 \cdot 4399557378$	
	$4 \cdot 0175320573$	
	$3 \cdot 4439955738$	
	5735364835	
	5510392918	
	224971917	
	206639734	
	18332183	
	13775982	
	4556201	
	4132795	
	423406	
	413279	
	10127	
	6888	
	3239	
	2755	
	484	
	482	
	2	

Since β is negative the two other roots are real, and since $\sqrt{-\beta} = \cdot 0081967265$, these roots will differ only by a unit in the second decimal figure; hence

$$a + \sqrt{-\beta} = 3 \cdot 2213244809 + \cdot 0081967265 = 3 \cdot 2295212074,$$
$$a - \sqrt{-\beta} = 3 \cdot 2213244809 - \cdot 0081967265 = 3 \cdot 2131277544,$$

and the negative root has been found $= -17 \cdot 4426489618$.

12. From the solution of the preceding example it is obvious that if the proposed equation has two *equal* roots, the value of β will be found to vanish entirely, and thus the method here employed will not only find the values of the imaginary roots of equations, but also the values of the real roots, whether they be equal, unequal, or nearly equal.

13. Before entering upon the solution of Biquadratic Equations, it may be useful to give a simple process for obtaining the necessary equations for the solution of cubic equations.

Let $x^3 + ax^2 + bx + c = 0$ be a cubic equation, and let its three roots be represented by $-(2y+a)$, $y + \sqrt{z}$, $y - \sqrt{z}$; then forming the equation of which these expressions are the roots, and equating the coefficients of the same powers of the unknown quantity in both equations, we get

$$z + 3y^2 + 2ay = -b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$(2y + a)(y^2 - z) = \quad c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating z from these equations, we have

$$y^3 + ay^2 + \frac{a^2 + b}{4} y + \frac{ab - c}{8} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and if the proposed equation be $x^3 + bx + c = 0$, then (3) will be

$$y^3 + \frac{b}{4} y - \frac{c}{8} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These are exactly the equations we have already found, and by which the solution of cubic equations has been completely effected. The value of z may be found in the manner already pointed out for the determination of the value of β in the former method. If $z = 0$, the equation has equal roots, if z is positive the roots are all real, and if z is negative, then two of the roots are imaginary. We now proceed to the solution of biquadratic equations, which may, in all cases, be effected by the solution of a cubic equation, and the extraction of the square root.

II. BIQUADRATIC EQUATIONS.

14. Let the equation, divested of its second term, if necessary, be

$$x^4 + bx^2 + cx + d = 0 \qquad \text{(1)}.$$

Then making $m = 4$ in equations (6), and recollecting that in the expression (A), the coefficients c', d', *etc.*, are all zero, we have $f_3 a = 0$, and $f_4 a = 0$; hence

$$A\beta - C = 0 \qquad \text{(2)},$$

$$\beta^2 - B\beta + D = 0 \qquad \text{(3)}.$$

Eliminating β from these two equations, we get the relation

$$ABC - A^2D - C^2 = 0 \qquad \text{(4)}.$$

In equations (3) art. 7 let $m = 4$, then, since $a = 0$, we have

$$\begin{aligned} A &= 4a, \\ B &= 6a^2 + b \\ C &= 4a^3 + 2ba + c \\ D &= a^4 + ba^2 + ca + d. \end{aligned}$$

Substituting these values of A, B, C, D in the relation (4), gives

$$a^6 + \frac{b}{2}a^4 + \frac{b^2 - 4d}{16}a^2 - \frac{c^2}{64} = 0 \qquad \text{(5)}.$$

This equation is identical with that obtained by WARING, in the *Philosophical Transactions* for 1779. By means of this equation we can obtain all the *four* roots of a biquadratic equation, whether they be real or imaginary, or whether some of them be equal or nearly equal.

The last term of this equation being negative, indicates that *one* of the values of a^2 is real and positive, and therefore a will have *two* equal values with contrary signs, viz. $+a$ and $-a$. These two values of a will furnish two values of β by means of equation (2), or an equation of a simpler kind, which we shall now deduce from it. Since $A = 4a$ and $C = 4a^3 + 2ba + c$, it follows from the equation $A\beta - C = 0$, that

$$\beta = \frac{C}{A} = \frac{4a^3 + 2ba + c}{4a} = a^2 + \frac{b}{2} + \frac{c}{4a} \qquad \text{(6)}.$$

This expression will readily give the value of β, since a^2 and a have been already determined, and we have only to effect the division of $\frac{1}{4}c$ by a, in order to find the values of β.

If the proposed equation is of the form $x^4 + ax^3 + bx^2 + cx + d = 0$, the resulting equation in a is

$$a^6 + \frac{3}{2}aa^5 + \frac{3a^2 + 2b}{4}a^4 + \frac{a(a^2 + 4b)}{8}a^3 + \frac{a(2ab + c) + b^2 - 4d}{16}a^2$$

$$+ \frac{a(ac + b^2 - 4d)}{32}a + \frac{abc - a^2d - c^2}{64} = 0 \qquad \text{(7)}.$$

15. The same results may be obtained very simply in the following manner.

Let $y \pm \sqrt{z_1}$ and $-y \pm \sqrt{z_2}$ be the four roots of the biquadratic equation

$$x^4 + bx^2 + cx + d = 0;$$

then forming the equation of which the expressions $y \pm \sqrt{z_1}$ and $y \pm \sqrt{z_2}$ are the roots, and equating the coefficients of the same powers of the unknown quantity, in both equations, we have

$$2y^2 + z_1 + z_2 = -b \qquad \text{(8)},$$

$$2y(z_1 - z_2) = -c \qquad \text{(9)}.$$

$$y^4 - y^2(z_1 + z_2) + z_1z_2 = d \qquad \text{(10)}.$$

From these three equations eliminate z_1 and z_2, and there results

$$y^6 + \frac{b}{2} y^4 + \frac{b^2 - 4d}{16} y^2 - \frac{c^2}{64} = 0 \quad \text{.........................(11).}$$

Hence y is known by the solution of a cubic, and then the values of z_1 and z_2 will be found from (8) and (9).

EXAMPLE I.

Solve completely the biquadratic equation

$$x^4 + x^3 + 4x^2 - 4x + 1 = 0 \quad \text{...............................(1).}$$

Here we must first divest the equation of its second term, by increasing the roots by ·25.

1 + 1	+ 4	— 4	+ 1	(— ·25
— ·25	— ·1875	— ·953125	1·23828125	
·75	3·8125	— 4·953125	2·23828125	
— ·25	— ·125	— ·921875		
·50	3·6875	— 5·875		
— ·25	— ·0625			
·25	3·6250			
— ·25				
0				

The equation in $x + ·25$ or x' is therefore

$$x'^4 + 3·625\, x'^2 - 5·875\, x' + 2·23828125 = 0 \quad \text{..........................(2);}$$

and with these coefficients for the values of b, c, d ; equation (5) is

$$a^6 + 1·8125\, a^4 + ·26171875\, a^2 - ·539306640625 = 0 \quad \text{.......................(3).}$$

This is a cubic equation in a^2, and it has only one real positive root, which is obtained in the usual manner, thus :

1 + 1·8125	+ ·26171875	— ·539306640625	(·43507305096 = a^2.
4	·88500	·458687500	
2·2125	1·14671875	— ·080619140625	
4	1·04500	684898125	
2·6125	2·19171875	— 12129328125	
4	91275	1195353125	
3·0125	2·28299375	— 175796875	
3	92175	168454089	
3·0425	2·37516875	— 7342786	
3	155375	7220143	
3·0725	2·39070625	— 122643	
3	155625	120336	
3·1025	2·4062687 5	— 2307	
5	2182 3	2166	
3·1075	2·4064869 8	— 141	
5	2182 3		
3·1125	2·406705 2		
5	9 3		
3·11175	2·406714 5		
	9 3		
	24,0,6,7,2,4		

$a^2 =$ ·43507305096 (·65960067538	$\pm$ ·65960067538) — 1·46875000000 ($\mp$ 2·2267260402
36	1·31920135076
125) 750	14954864924
625	13192013508
1309) 12573	1762851416
11781	1319201351
13186) 79205	443650065
79116	395760405
131,9,2,0,0) 890960	47889660
791520	46172047
99440	1717613
92344	1319201
7096	398412
6596	395760
500	2652
396	2638
104	14
104	13

Hence $a = \pm$ ·65960067538, and the rational parts of the *four* roots are $-$·25 $\pm$ ·65960067538, or ·40960067538 and $-$·90960067538.

The values of β are found from the formula $\beta = a^2 + \frac{b}{2} + \frac{c}{4a}$.

$a^2 =$	·4350730509	$a^2 =$	·43507305096
$\frac{b}{2} =$	1·8125	$\frac{b}{2} =$	1·8125
$+\frac{c}{4a} =$	$-$ 2·2267260402	$-\frac{c}{4a} =$	2·2267260402
$\beta =$	·0208470107	$\beta =$	4·4742990912

The two values of β being both positive, indicate that *all the four roots of the proposed equation are imaginary*, and the roots themselves are found to be

$$·40960067538 \pm \sqrt{-\ ·0208470107},$$

$$-·90960067538 \pm \sqrt{-4·4742990912}.$$

In this example the operation has been put down at full length, to show the amount of labour required to determine the four roots, whether real or imaginary, of a biquadratic equation; but in the subsequent examples, the common operations of division and the extraction of the square root will only be indicated by the usual signs.

EXAMPLE II.

Solve completely the equation $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$.

(*Prof. Young's Math. Dissertations*, p. 160.)

1 — 80	+ 1998	— 14937	+ 5000	(20
20	— 1200	15960	20460	
— 60	798	1023	25460	
20	— 800	— 40		
— 40	— 2	983		
20	—400			
— 20	—402			
20				
0				

Hence $b = -402$, $c = 983$ and $d = 25460$; consequently eq. (5) becomes

$$a^6 - 201\,a^4 + 3735{\cdot}25a^2 - 15098{\cdot}265625 = 0.$$

This equation in a^2 has three real roots, and they are all positive. It will suffice to find *any one* of these roots, and we shall obtain from it two values of a, and thence two values of β from the formula (5).

1 — 201	+ 3735·25	— 15098·265625 (5·80022839388 = a^2
5	— 980	13776·25
— 196	2755·25	— 1322·015625
5	— 955	1321·672
— 191	1800·25	— ·343625000
5	— 148·16	·300906656
— 1860	1652·09	— 42718344
8	— 147·52	30089858
— 1852	1504·570000	— 12628486
8	— 3672	12035902
— 1844	1504·53328	— 592584
8	— 3672	451346
— 18 36 00	1504·4965 6	— 141238
	— 36 7	135404
	1504·4928 9	— 5834
	— 36 7	4513
	1504·489 2	— 1321
	— 1 4	1203
	1504·487 8	— 118
	— 1 4	
	15,0,4,4,8,6	

Hence $a = \sqrt{5{\cdot}80022839388} = \pm 2{\cdot}4083663330$, and consequently $20 \pm 2.408366333 = 22{\cdot}408366333$ and $17{\cdot}591633667$, which are the rational parts of the four roots. Again

$$\begin{aligned} a^2 &= \quad 5{\cdot}80022839 \\ \frac{b}{2} &= -201 \\ \frac{c}{4a} &= \quad 102{\cdot}04012431 \\ \hline \beta &= -\;93{\cdot}15964730 \end{aligned} \qquad\qquad \begin{aligned} a^2 &= \quad 5{\cdot}80022839 \\ \frac{b}{2} &= -201 \\ -\frac{c}{4a} &= -102{\cdot}04012431 \\ \hline \beta &= -297{\cdot}23989592 \end{aligned}$$

Since the values of β are both *negative*, the equation has four real roots, and these roots are found to be

$$22{\cdot}408366333 \pm \sqrt{93{\cdot}15964730},$$
$$17{\cdot}591633667 \pm \sqrt{297{\cdot}23989592}.$$

Extracting the square roots of the irrational parts of these numbers, we obtain the four following positive roots of the proposed equation; viz.

$$\begin{aligned} 22{\cdot}408366333 + \;\; 9{\cdot}651924539 &= 32{\cdot}060290872 \\ 22{\cdot}408366333 - \;\; 9{\cdot}651924539 &= 12{\cdot}756441794 \\ 17{\cdot}591633667 + 17{\cdot}240646621 &= 34{\cdot}832280288 \\ 17{\cdot}591633667 - 17{\cdot}240646621 &= \;\; 0{\cdot}350987046 \\ \hline \text{Proof} \;\; . \;\; . &\quad 80{\cdot}000000000 \end{aligned}$$

EXAMPLE III.

Solve completely the equation

$$x^4 + 312x^3 + 23337x^2 - 14874x + 2360 = 0.$$

Increase the roots of the equation by 78, to divest it of the second term.

1 + 312	+ 23337	— 14874	+ 2360 (—78
— 70	— 16940	— 447790	32386480
242	6397	— 462664	32388840
— 70	— 12046	395010	— 289168
172	— 5643	— 67654	32099672
— 70	— 7140	103800	
102	— 12783	36146	
— 70	— 192	104824	
32	— 12975	140970	
— 8	— 128		
24	— 13103		
— 8	— 64		
16	— 13167		
— 8			
8			
— 8			
0			

The coefficients of the transformed equation are

$$b = -13167,$$
$$c = 140979,$$
$$d = 32099672.$$

The equation in a^2 is consequently

$$a^6 - 6583{\cdot}5a^4 + 2810700{\cdot}0625a^2 - 310508451{\cdot}5625 = 0,$$

which has three real positive roots. We shall develope one of them.

1 —6583·5	+ 2810700·0625	— 310508451·5625	(225 = a^2;
200	— 1276700	306800012·5	
—6383·5	1534000·0625	— 3708439·0625	
200	— 1236700	3560601·25	$\therefore a = \pm 15$.
—6183·5	297300·0625	— 147837·8125	
200	— 119270	147837·8125	
—5983·5	178030·0625		
20	— 118870		
—5963·5	59160·0625		
20	— 29592·5		
—5943·5	29567·5625		
20			
—5923·5			
5			
—5918·5			

Hence the rational parts of the four roots of the proposed equation are $-78 \pm 15 = -63$ and -93; and therefore we have

$$\beta = a^2 + \frac{b}{2} + \frac{c}{4a} = 225 - 6583{\cdot}5 + 2349{\cdot}5 = -4009;$$

$$\beta = a^2 + \frac{b}{2} - \frac{c}{4a} = 225 - 6583{\cdot}5 - 2349{\cdot}5 = -8708.$$

Consequently all the four roots of the equation are real, and they are

$$\begin{aligned} -63+\sqrt{4009} &= \cdot 3166644731069, \\ -63-\sqrt{4009} &= -126\cdot 3166644731069, \\ -93+\sqrt{8708} &= \cdot 3166651783056, \\ -93-\sqrt{8708} &= -186\cdot 3166651783056. \end{aligned}$$

This remarkable equation was sent to Professor Young, of Belfast, by the venerable Mr. Lockhart, a gentleman who has laboured long and successfully in this department of science. In his neat little treatise on "*The Analysis and Solution of Cubic and Biquadratic Equations,*" Professor Young has analysed this equation with his usual ability, and determined the roots correctly as far as the ninth decimal figure inclusive. It is a remarkable circumstance that the method of solution here developed is admirably calculated to determine the roots of equations not only when they are real and imaginary, but also when they are in the form of binomial surds.

III. EQUATIONS OF THE FIFTH DEGREE.

16. Let the equation of the fifth degree, divested of its second term, if necessary, be

$$x^5+bx^3+cx^2+dx+e=0 \quad\ldots\ldots\ldots\ldots(1).$$

Make $m=5$ in equations (6); then since d', e', f', *etc.*, in the expression (A) are all zero, we have $f_5 a=0$, and $f_6 a=0$; therefore we obtain

$$\beta^2-\mathrm{B}\beta+\mathrm{D}=0 \quad\ldots\ldots\ldots\ldots(2),$$

$$\mathrm{A}\beta^2-\mathrm{C}\beta+\mathrm{E}=0 \quad\ldots\ldots\ldots\ldots(3).$$

Eliminating β from these two equations, we get the relation

$$(\mathrm{AB}-\mathrm{C})(\mathrm{CD}-\mathrm{BE})=(\mathrm{AD}-\mathrm{E})^2 \quad\ldots\ldots\ldots\ldots(4).$$

Also in equations (3) art. 7 let $m=5$, then since $a=0$,

$$\begin{aligned} \mathrm{A} &= 5a \\ \mathrm{B} &= 10a^2+b \\ \mathrm{C} &= 10a^3+3ba+c \\ \mathrm{D} &= 5a^4+3ba^2+2ca+d \\ \mathrm{E} &= a^5+ba^3+ca^2+da+e. \end{aligned}$$

Substituting these values of A, B, C, D, E in the relation (4) gives the equation

$$a^{10}+\frac{3b}{4}a^8+\frac{c}{8}a^7+\frac{3(b^2-d)}{16}a^6+\frac{2bc-11e}{32}a^5+\frac{b(b^2-2d)-c^2}{64}a^4+\frac{c(b^2-4d)-4be}{128}a^3$$

$$+\frac{d(b^2-4d)-c(bc-7e)}{256}a^2-\frac{e(b^2-4d)+c^3}{512}a-\frac{\{c(cd-be)+e^2}{1024}=0 \quad\ldots\ldots\ldots\ldots(5).$$

If this equation be transformed into another whose roots are *twice* those of the former, the transformed equation will be

$$a^{10}+3ba^8+ca^7+3(b^2-d)a^6+(2bc-11e)a^5+\{b(b^2-2d)-c^2\}a^4+\{c(b^2-4d)-4be\}a^3$$

$$+\{d(b^2-4d)-c(bc-7e)\}a^2-\{e(b^2-4d)+c^3\}a-c(cd-be)-e^2=0 \quad\ldots\ldots\ldots\ldots(5'),$$

which may sometimes be more convenient in practice than equation (5).

The elimination of a from the equations (2) and (3) would lead to an equation of the tenth degree, analogous to the complicated and unmanageable equation of Waring, and which is given by Lagrange at p. 111 of his "*Traité des Résolutions des Equations Numeriques.*"

Having obtained from equation (5) the values of a, the values of β will be found from the equations (2) and (3); but those values of β which are the *same* in both equations are only to be taken and the others rejected. In order to

avoid any ambiguity in determining the values of β, it will therefore be necessary to find the greatest common measure of (2) and (3), and equate it to zero, the equation thus obtained will furnish the proper values of β. Dividing (3) by (2) then, we get as a remainder, the expression

$$(\mathrm{AB}-\mathrm{C})\beta-(\mathrm{AD}-\mathrm{E}).$$

Now this expression being of the first degree in β, is necessarily the greatest common measure of (2) and (3). Equating it to zero, we get

$$\left.\begin{array}{l}\beta=\dfrac{\mathrm{AD}-\mathrm{E}}{\mathrm{AB}-\mathrm{C}}=\dfrac{24a^5+14ba^3+9ca^2+4da-e}{40a^3+2ba-c} \\[2ex] \text{or}\quad \beta=\dfrac{3}{5}a^2+\dfrac{8b}{25}+\dfrac{240ca^2-4(b^2-25d)a+8bc-25e}{25(40a^3+2ba-c)}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This equation will determine the value of β, when that of a is known, and thus the roots of the proposed equation will be obtained.

17. In the solution of equations of the fifth degree, it will be sufficient to find *one* of the real roots of the given equation, and also one of the real roots of equation (5); for if r denote the real root of (1), and $a_1\pm\sqrt{-\beta_1}$, and $a_2\pm\sqrt{-\beta_2}$, the other four roots, then, since the second term of the equation is absent, we have

$$r+2a_1+2a_2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

an equation which will give a_2, when r and a_1 are determined.

It is worthy of remark that, if the values of the coefficients b, c, d, e be such as to render the last term of eq. (5) zero, then *one* root of this equation will be 0; and *all* the roots of the proposed equation can be obtained without much additional labour. Thus if $a_1=0$; then the form of two of the roots will be $\pm\sqrt{-\beta_1}$, where

$$\beta_1=\frac{e}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

EXAMPLE I.

Solve completely the equation $\quad x^5+x^4+x^3-2x^2+2x-1=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

This equation has one real positive root between 0 and 1, since the last term is negative, and we shall divest the equation of its second term, and then continue the operation for determining the real root.

<table>
<tr><td>1 + 1.0</td><td>+ 1·00</td><td>— 2·000</td><td>+ 2·0000</td><td>— 1·00000</td><td>(— .2</td></tr>
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<tr><td>— ·2</td><td>— ·12</td><td>— 144</td><td>4624</td><td>139008</td><td>·6407459688 = root.</td></tr>
<tr><td>·6</td><td>·72</td><td>— 2·312</td><td>2·8960</td><td>— 966400000</td><td></td></tr>
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<tr><td>— ·2</td><td>— 4</td><td>992</td><td>4544</td><td>17918179</td><td></td></tr>
<tr><td>·2</td><td>·60</td><td>— 1448</td><td>219200000</td><td>— 1178397</td><td></td></tr>
<tr><td>— ·2</td><td>64</td><td>2016</td><td>17625856</td><td>1025378</td><td></td></tr>
<tr><td>0·8</td><td>124</td><td>568</td><td>236825856</td><td>— 153019</td><td></td></tr>
<tr><td>8</td><td>128</td><td>3552</td><td>18797824</td><td>128183</td><td></td></tr>
<tr><td>16</td><td>252</td><td>4120000</td><td>2556236|8,0</td><td>— 24836</td><td></td></tr>
<tr><td>8</td><td>192</td><td>286464</td><td>3503|0</td><td>23073</td><td></td></tr>
<tr><td>24</td><td>444</td><td>4406464</td><td>2559739|8</td><td>— 1763</td><td></td></tr>
<tr><td>8</td><td>256</td><td>292992</td><td>3506|7</td><td>1538</td><td></td></tr>
<tr><td>32</td><td>70000</td><td>4699456</td><td>256324|6</td><td>— 225</td><td></td></tr>
<tr><td>8</td><td>1616</td><td>299584</td><td>20|0</td><td>205</td><td></td></tr>
<tr><td>404</td><td>71616</td><td>499|90,40</td><td>256344|6</td><td>— 20</td><td></td></tr>
<tr><td>4</td><td>1632</td><td>|53</td><td>20|</td><td>20</td><td></td></tr>
<tr><td>408</td><td>73248</td><td>500|43</td><td>25636|5</td><td></td><td></td></tr>
<tr><td>4</td><td>1648</td><td>|53</td><td>|2</td><td></td><td></td></tr>
<tr><td>412</td><td>74896</td><td>500 96</td><td>25636|7</td><td></td><td></td></tr>
<tr><td>4</td><td>1644</td><td>|53</td><td>|2</td><td></td><td></td></tr>
<tr><td>416</td><td>76,560</td><td>1·5|01</td><td>25,6,3,7</td><td></td><td></td></tr>
<tr><td>4</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>1.420</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

The real root of (1) is therefore ·6407459688, and the coefficients of the equation divested of its second term are

$$b = \cdot 6,\quad c = -2\cdot 44,\quad d = 2\cdot 896,\ \text{ and } e = -1\cdot 48672;$$

$\therefore$ by (5), $a^{10} + \cdot 45a^8 - \cdot 305a^7 - \cdot 4755a^6 + \cdot 41956a^5 - \cdot 14395a^4 + \cdot 2418335a^3 - \cdot 04173315a^2$
$- \cdot 004219065a - \cdot 0168705116 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

Supplying the absent term of this equation with a cipher-coefficient, there will be seven variations of signs; hence by Budan's criterion, the equation

In $(a - \cdot 6)$ gives the signs $+ + + + + + + + + + +$ 7 variations *lost*,

" $\left(\frac{1}{a} - \frac{1}{\cdot 6}\right)$ .... $+ + + + + + + + + + -$ 1 variation *left*.

The equation (2) has therefore *six* imaginary roots in the interval between 0 and ·6. Change the signs of the alternate terms, then the equation

In $(-a)$ gives the signs $+ + + + - - - - - + -$ 3 variations,

" $(-a - 1)$ .... $+ + + + + + + + + + +$ 3 variations *lost*.

" $\left(-\frac{1}{a} - 1\right)$ .... $+ + + + + + + + + + -$ 1 variation *left*.

The equation (2) has therefore *two* imaginary roots in the interval 0 and — 1; and it has consequently only *two real roots*, the one positive and the other negative. We shall find the positive root of equation (2).

<table>
<tr><td>1</td><td>+0</td><td>+·45</td><td>—·305</td><td>—·4755</td><td>+·41956</td><td>—·143950</td><td>+·2418335</td><td>—·04173315</td><td>—·004219065</td><td>—·0168705116(·517041280</td></tr>
<tr><td></td><td>5</td><td>25</td><td>350</td><td>225</td><td>— 22650</td><td>96530</td><td>— 237100</td><td>10906175</td><td>33664300</td><td>+ 147226175</td></tr>
<tr><td></td><td>5</td><td>70</td><td>45</td><td>— 4530</td><td>19306</td><td>— 47420</td><td>2181235</td><td>6732860</td><td>29445235</td><td>— 21478941</td></tr>
<tr><td></td><td>5</td><td>50</td><td>600</td><td>3225</td><td>— 6525</td><td>63905</td><td>82425</td><td>11318300</td><td>90255800</td><td>12335396</td></tr>
<tr><td></td><td>10</td><td>120</td><td>645</td><td>— 1305</td><td>12781</td><td>16485</td><td>2263660</td><td>18051160</td><td>11970103|5</td><td>—9143545</td></tr>
<tr><td></td><td>5</td><td>75</td><td>975</td><td>8100</td><td>33975</td><td>233780</td><td>1251325</td><td>17574925</td><td>365292|1</td><td>9088789</td></tr>
<tr><td></td><td>15</td><td>195</td><td>1620</td><td>6795</td><td>46756</td><td>250265</td><td>3514985</td><td>356260|85</td><td>12335395|6</td><td>— 54756</td></tr>
<tr><td></td><td>5</td><td>100</td><td>1475</td><td>15475</td><td>111350</td><td>790530</td><td>5203975</td><td>9031|27</td><td>374644|4</td><td>53058</td></tr>
<tr><td></td><td>20</td><td>295</td><td>3095</td><td>22270</td><td>158106</td><td>1040795</td><td>8718|960</td><td>365292|12</td><td>1271004|0</td><td>— 1698</td></tr>
<tr><td></td><td>5</td><td>125</td><td>2100</td><td>25975</td><td>241225</td><td>1996655</td><td>312|311</td><td>9352|3</td><td>27394|5</td><td>1327</td></tr>
<tr><td></td><td>25</td><td>420</td><td>5195</td><td>48245</td><td>399331</td><td>303|7450</td><td>9031|271</td><td>374644|4</td><td>1298398|5</td><td>— 371</td></tr>
<tr><td></td><td>5</td><td>150</td><td>2850</td><td>40225</td><td>442350</td><td>8|5658</td><td>321|03</td><td>9682|2</td><td>27885|9</td><td>265</td></tr>
<tr><td></td><td>30</td><td>570</td><td>8045</td><td>88470</td><td>8|41681</td><td>312|3108</td><td>9352|30</td><td>3843|27</td><td>13262|8|4</td><td>— 106</td></tr>
<tr><td></td><td>5</td><td>175</td><td>3725</td><td>58850</td><td>|14898</td><td>8|7164</td><td>329|89</td><td>70|2</td><td>1|6|</td><td>106</td></tr>
<tr><td></td><td>35</td><td>745</td><td>11770</td><td>147320</td><td>8|56579</td><td>321|027</td><td>9682|2</td><td>3913|5</td><td>13264|4</td><td></td></tr>
<tr><td></td><td>5</td><td>200</td><td>4725</td><td>1661</td><td>|15065</td><td>8|868</td><td>338|9</td><td>70|2</td><td>1|6</td><td></td></tr>
<tr><td></td><td>40</td><td>945</td><td>16495</td><td>14898,1</td><td>8|7164</td><td>329|89</td><td>10|021</td><td>3983|7</td><td>13,2,6,6</td><td></td></tr>
<tr><td></td><td>5</td><td>225</td><td>117</td><td>167'3</td><td>|1523</td><td>9|02</td><td></td><td>70|2</td><td></td><td></td></tr>
<tr><td></td><td>45</td><td>1170</td><td>1661,2</td><td>15065</td><td>8|8687</td><td>338|9</td><td></td><td>|40|54</td><td></td><td></td></tr>
<tr><td></td><td>5</td><td>5</td><td>12</td><td>168</td><td>|1540</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>5,0</td><td>117,5</td><td>1673</td><td>15233</td><td>9|0227</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>118</td><td>1685</td><td>15401</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Hence $a_1 = \cdot 517041280$ and $r = .8407459688$; therefore by formula (7) we have

$$r + 2a_1 + 2a_2 = 0;\ \text{ or }\ a_2 = -\frac{r}{2} - a_1 = -\cdot 9374142644.$$

Now by the second of the formulas (6) we have

$$\beta = \frac{3}{5}a^2 + .192 + \frac{-23\cdot 424a^2 + 11\cdot 3536a + 1\cdot 01824}{40a^3 + 1\cdot 2a + 2\cdot 44};$$

hence if $a = \cdot 517041280$, the value of β will be ·4253434585,

if $a = -\cdot 9374142644$, ... 1·6741603770;

consequently the five roots of the proposed equation are

$$\cdot 6407459688$$

$$\cdot 3170412800 \pm \sqrt{-\cdot 4253434585}$$

$$-1\cdot 1374142644 \pm \sqrt{-1\cdot 6741603770}.$$

EXAMPLE II.

Solve completely the equation

$$x^5 - 32x^3 + 72x^2 - 185x + 360 = 0 \ldots\ldots\ldots\ldots (1).$$

By changing the signs of the alternate terms of this equation it is evident that there is only variation of sign, and therefore the real root is negative.

1 — 0	— 32	— 72	— 185	— 360	(6·88855039
6	36	24	— 288	—2838	
6	4	— 48	— 473	—3198·00000	
6	72	456	2448	2728 23168	
12	76	408	19750000	— 46976832	
6	108	1104	14352896	42270253	
18	184	1512000	34102896	— 4706579	
6	144	282112	16771584	4402501	
24	32800	1794112	5087448 0	— 304078	
6	2464	302336	196333 6	276231	
308	35264	2096448	528378 16	— 27847	
8	2528	323072	199123 1	27630	
316	37792	24195 20	548290 5	— 217	
8	2592	346 50	2022 1	166	
324	40384	24541 70	550312 6	— 51	
8	2656	348 69	2024 9	50	
332	43 040	24890 39	55233 7	1	
8	273	350 88	12 6		
.3408	43 313	252 41	55246 3		
	273	35	12 6		
	43 586	252 76	5525 9		
	273	35	1		
	43 86	253 11	5,5,2,6		
	27	35			
	·441	·2153			

The negative root of the equation is $-6·88855039$, and to form the equation in α we have $b = -32$, $c = 72$, $d = -185$, and $e = 360$. If we compute the last or absolute term of the equation (5) we get (since $e = 5c$),

$$c(cd - be) + e^2 = 72^2(-185 + 160 + 25) = 0;$$

consequently one root of the equation (5) is $\alpha_1 = 0$, and from the equation (7) we have

$$\alpha_2 = -\frac{r}{2} = 3·444275195.$$

Hence by (6) we get the following values of β_1 and β_2, namely

$$\beta_1 = \frac{e}{c} = \frac{360}{72} = 5 \text{ ; and}$$

$$\beta_2 = \frac{3}{5}\alpha_2^2 - \frac{256}{25} + \frac{27}{50}\cdot\frac{160\alpha_2^2 - 323\alpha_2 - 254}{5\alpha_2^3 - 8\alpha_2 - 9} = -1·4109051373.$$

The equation has only two imaginary roots, and the five roots are found to be

$$-6·88855039;\ +\sqrt{-5};\ -\sqrt{-5};$$
$$-3·444275195 + \sqrt{1·4109051373} = 4·632090474;$$
$$-3·444275195 - \sqrt{1·4109051373} = 2·256459916.$$

This example has been selected for the purpose of showing the remarkable facility of the method, in determining the roots of equations when some of them are of the form $\pm\sqrt{-\beta}$, whether the value of β be positive or negative. Example 8, p. 161, of Professor YOUNG's *Mathematical Dissertations*, is an equation of this nature, the two imaginary roots being $\pm\sqrt{-1}$, and the three real roots are the same as those of the equation which has been solved above.

IV. EQUATIONS OF THE SIXTH DEGREE.

18. Let the proposed equation be

$$x^6 + bx^4 + cx^3 + dx^2 + ex + f = 0 \ldots\ldots\ldots\ldots(1).$$

Then making $m = 6$ in equations (6), and recollecting that e', f', *etc.* are zero in the expression ($\mathcal{A}$), we get $f_5 a = 0$ and $f_6 a = 0$; consequently

$$\mathrm{A}\beta^2 - \mathrm{C}\beta + \mathrm{E} = 0 \ldots\ldots\ldots\ldots(2),$$

$$\beta^3 - \mathrm{B}\beta^3 + \mathrm{D}\beta - \mathrm{F} = 0 \ldots\ldots\ldots\ldots(3).$$

Eliminating β from these two equations, we get the relation

$$\{\mathrm{C}(\mathrm{AB} - \mathrm{C}) - \mathrm{A}(\mathrm{AD} - \mathrm{E})\}.\{\mathrm{E}(\mathrm{AD} - \mathrm{E}) - \mathrm{ACF}\} = \{\mathrm{E}(\mathrm{AB} - \mathrm{C}) - \mathrm{A}^2\mathrm{F}\}^2 \ldots\ldots\ldots\ldots(4).$$

In equations (3) art. 7, let $m = 6$, then, since $a = 0$, we get

$$\begin{aligned}
\mathrm{A} &= 6a \\
\mathrm{B} &= 15a^2 + b \\
\mathrm{C} &= 20a^3 + 4ba + c \\
\mathrm{D} &= 15a^4 + 6ba^2 + 3ca + d \\
\mathrm{E} &= 6a^5 + 4ba^3 + 3ca^2 + 2da + e \\
\mathrm{F} &= a^6 + ba^4 + ca^3 + da^2 + ea + f.
\end{aligned}$$

Substituting these values of A, B, C, D, E, F in the relation (4), cancelling the equal terms in both members, and arranging the result, we get the equation

$$\begin{aligned}
&a^{15} + ba^{13} + \frac{c}{4}a^{12} + \frac{3b^2 - d}{8}a^{11} + \frac{3bc - 5e}{16}a^{10} + \frac{4b(2b^2 - d) + 11f}{128}a^9 + \frac{3c(b^2 - d) - 6be}{64}a^8 \\
&+ \frac{b^2(b^2 + 2d) - 3(2bf + ce) - 7d^2}{256}a^7 + \frac{8\{bc(b^2 - 2d) - e(b^2 + 5d) - c^3\} + 75cf}{2048}a^6 \\
&+ \frac{(4bd - 27f)(b^2 - 3d) + 12c(be - cd) - 24e^2}{2048}a^5 + \frac{8\{cd(b^2 - 4d) - c^2(bc - 3e) - bde\} + 3f(5bc + 33ef)}{8192}a^4 \\
&+ \frac{4\{d^2(b^2 - 4d) - c^2(2bd + c^2) + be(bc - 5e) + 7cde\} - f\{8b(2b^2 - 9d) - 3(c^2 - 36f)\}}{16384}a^3 - \frac{c(c^2d - 3e^2)}{4096}a^2 \\
&- \frac{e^2(b^2 - 3d) + c^2(d^2 + ce) - bcde + 3cf(bc - 3e)}{16384}a + \frac{e^2(bc - e) - c^2(de - cf)}{32768} = 0 \ldots\ldots\ldots\ldots(5).
\end{aligned}$$

If the values of the coefficients b, c, d, e, f be such as to render the last term of the equation zero, then we shall have $a_1 = 0$; and if the last *two* terms vanish, then $a_1 = 0$ and $a_2 = 0$.

To determine the value of β, multiply eq. (3) by A, and divide by $\mathrm{A}\beta^2 - \mathrm{C}\beta + \mathrm{E}$; then after two divisions, the remainder will be found to be

$$\{\mathrm{A}(\mathrm{AD} - \mathrm{E}) - \mathrm{C}(\mathrm{AB} - \mathrm{C})\}\beta + \mathrm{E}(\mathrm{AB} - \mathrm{C}) - \mathrm{A}^2\mathrm{F}.$$

Equating this to zero, gives

$$\beta = \frac{\mathrm{E}(\mathrm{AB} - \mathrm{C}) - \mathrm{A}^2\mathrm{F}}{\mathrm{C}(\mathrm{AB} - \mathrm{C}) - \mathrm{A}(\mathrm{AD} - \mathrm{E})} \ldots\ldots\ldots\ldots(6);$$

or, substituting for A, B, C, D, E, F their values given above, we get

$$\beta = \frac{3}{7}a^2 + \frac{1408ba^6 + 1296ca^5 + 32(b^2 + 25d)a^4 + 20(bc + 11e)a^3 + 2(14bd - 9c^2 - 126f)a^2 + 11(be - cd)a - 7ce}{7\{896a^6 + 128ba^4 - 40ca^3 + 8(b^2 - 3d)a^2 - 2(bc - 3e)a - c^2\}} \ldots\ldots(7).$$

If $a_1 = 0$, then by (7) $\beta_1 = \frac{e}{c}$, and the proposed equation may readily be depressed to one of the fourth degree, and the complete solution effected by means of a cubic equation.

We might here have solved an equation of the sixth degree, but enough has been done in previous examples to show the application of the method to any numerical equation.

NOTE A.

In the preceding paper we have given a simple method of finding all the roots of a cubic equation; but there is another method of solution which it may be useful to advert to, inasmuch as all the three roots may be found *simultaneously*.

Let $a + \sqrt{-\beta}$, $a - \sqrt{-\beta}$ and r be the three roots of the cubic equation $x^3 + ax^2 + bx + c = 0$; then if we form the equation of which these are the roots, it will be

$$x^3 - (2a + r)x^2 + (a^2 + 2ra + \beta)\, x - r\,(a^2 + \beta) = 0 \dots\dots\dots\dots (1).$$

Reduce the roots of this equation by a, the rational part of the roots $a \pm \sqrt{-\beta}$, and we have the following operation:

$$\begin{array}{rrrl}
1 & -(2a + r) & +(a^2 + 2ra + \beta) & -r(a^2 + \beta)\big(a \\
 & \underline{a} & \underline{-a^2 - ra} & \underline{+ra^2 + a\beta} \\
 & -a - r & ra + \beta & \beta(a - r) \\
 & \underline{a} & \underline{-ra} & \\
 & -r & \beta & \\
 & \underline{a} & & \\
 & a - r & &
\end{array}$$

Hence the transformed equation in $x - a$ or x' is

$$x'^3 + (a - r)x'^2 + \beta x' + \beta(a - r) = 0 \dots\dots\dots\dots (2);$$

and the coefficients of the several powers of the unknown quantity have the relation which has been given in art. 9, pp. 8 and 9. Now it is obvious that if the roots of the proposed equation $x^3 + ax^2 + bx + c = 0$, be reduced by a, *the coefficient of the first power of the unknown quantity in the transformed equation will be the value of* β; and since by the theory of equations we have

$$r + 2a = -a, \text{ or } a = -\tfrac{1}{2}(a + r) \dots\dots\dots\dots (3);$$

the values of all the three quantities r, a and β may be obtained simultaneously, by combining the operation for finding the value of the real root with that for reducing the roots by a, as is evident from the preceding transformation.

EXAMPLE.

Find all the roots of the cubic equation $x^3 + 10\,x^2 + 5x - 2600 = 0$.

This equation has one real *positive* root, and the rational part of the two other roots is *negative;* hence writing the coefficients of x^2 and x in duplicate, and changing the sign of the coefficient of the second term, to avoid operating with the negative value of a, we have the subjoined operation.

```
                                                                          10
1 + 10     + 5           — 10            + 5                    — 2600( 11·00679933972 = r
    11       231            10               0                     2596( 10·50339966986 = — a
    21       236             0               5                    — 4
    11       352            10             100                      3529548216
    32       5880000        10             105                    — 470451784
    11           258036     10              10·25                   411982355
 43006       588258036     205              10·50                 — 58469429
     6           258072    210                   64509               52972218
 43012       58851610|8    21503                 64518             — 5497211
     6           3011|3    21506                  645279              5297260
4|30|18      58854622|1    215093                 645288            — 199951
                 3011|3    215096                 19358991            176575
             5885763|3     2150999                19359072          — 23376
                  38|7     2151008                 193591611          17657
             5885802|0     21510179                193591692        — 5719
                  38|7     21510188                 25812237           5297
             588584|1      215101976                 2581224        — 422
                   |4              2                  387184            412
             588584|5      430,2,0,3,9,5               34416          — 10
                   |4                                   2581
           5,8,8,5,8,5
```

$\beta = 125{\cdot}896220477245(11{\cdot}2203485007038 = \sqrt{\beta}$.

The three roots of the proposed cubic equation are consequently

$11{\cdot}00679933972$ and $-\,10{\cdot}50339966986 \pm 11{\cdot}2203485007038\,\sqrt{-1}$.

NOTE B.

Let $x^7 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$ be an equation of the seventh degree, and if two of its roots be of the form $\pm\sqrt{-\beta}$, then the value of a, the rational part of the roots, will be zero, and the values of A, B, C, D, E, F, G, given in equations (3) p. 7, will reduce to

$$\mathrm{A} = a,\ \mathrm{B} = b,\ \mathrm{C} = c,\ \mathrm{D} = d,\ \mathrm{E} = e, \mathrm{F} = f,\ \mathrm{G} = g.$$

Now if $m = 7$ in equations (6) art. 7; then $f_6 a = 0$, and $f_7 a = 0$, and we get

$$\beta^3 - b\beta^2 + d\beta - f = 0 \ldots\ldots(1),$$

$$a\beta^3 - c\beta^2 + e\beta - g = 0 \ldots\ldots(2).$$

Eliminating β from these two equations, we get the relation

$$\{(ab-c)(cf-bg)-(ad-e)(af-g)\}\{(ad-e)(ef-dg)-(af-g)(cf-bg)\} = \{(ab-c)(ef-dg)-(af-g)^2\}^2 \ldots (3).$$

Hence if the coefficients of the unknown quantity in the several terms of an equation of the seventh degree be such as to satisfy the relation (3), then the equation will have two roots of the form $\pm\sqrt{-\beta}$, and the value of β deduced from the simultaneous equations (1) and (2) will be

$$\beta = \frac{(ab-c)(ef-dg)-(af-g)^2}{(ab-c)(cf-bg)-(ad-e)(af-g)} = \frac{(ad-e)(ef-dg)-(af-g)(cf-bg)}{(ab-c)(ef-dg)-(af-g)^2} \ldots\ldots(4).$$

If the equation wants the second term, then $a = 0$, and (3) and (4) reduce to

$$(cf - bg + de)\{c(cf-bg) + eg\} + e^2 f(bc - e) = (cd - g)\{2cef - g(cd - g)\} \ldots\ldots(3'),$$

and, $$\beta = \frac{c(ef-dg) + g^2}{c(cf-bg) + eg} = \frac{e^2 f - g(cf - bg + de)}{cef - g(cd - g)} \ldots\ldots(4').$$

In the relation (3) let $g = 0$, then we shall have

$$(abc - a^2 d - c^2 + ae)(ade - acf - e^2) = (abe - a^2 f - ce)^2,$$

which by partial multiplication, and cancelling equal terms from both members, gives

$$(bc - ad + e)\{a(cf - de) + e^2\} + c^2(cf - de) = (be - af)\{a(af - be) + 2ce\} \ldots\ldots(5).$$

If $a = 0$, then (5) reduces to $$c^2(cf - de) = e^2(bc - e) \ldots\ldots(5').$$

Also the value of β, when $g = 0$ in equation (4), becomes

$$\beta = \frac{e(ab-c) - a^2 f}{c(ab-c) - a(ad-e)} = \frac{e(ad-e) - acf}{e(ab-c) - a^2 f} \ldots(6), \quad \text{and when } a = 0,\ \beta = \frac{e}{c} \ldots\ldots(6').$$

Hence if the equation $x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ be such that the coefficients a, b, c, *etc.*, satisfy the relation (5), then two of its roots will be of the form $\pm\sqrt{-\beta}$, and their value will be found from (6). If the equation wants the second term, and if the relation (5′) is satisfied, the equation will have two roots equal to $\pm\sqrt{-\frac{e}{c}}$.

Again, in the relation (5) and the formula for the value of β (6) let $f = 0$, then we get

$$(ab - c)(cd - be) = (ad - e)^2 \ldots\ldots(7),$$

and $$\beta = \frac{e(ab-c)}{c(ab-c) - a(ad-e)} = \frac{ad - e}{ab - c} \ldots\ldots(8).$$

When $a = 0$, then the formulas (7) and (8) reduce to

$$c(be - cd) = e^2 \ldots\ldots(7'), \quad \text{and } \beta = \frac{e}{c} \ldots\ldots(8').$$

Hence if the coefficients of the several terms of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ be such as to satisfy the relation (7), then the equation will have two roots of the form $\pm\sqrt{-\beta}$, and their value will be obtained from (8); and if the coefficients b, c, d, e of the terms of the equation $x^5 + bx^3 + cx^2 + dx + e = 0$ be such as to satisfy (7′), then two roots of the equation will be $\pm\sqrt{-\frac{e}{c}}$.

As an example, let the equation of the fifth degree be

$$x^5 - 36x^3 + 72x^2 - 37x + 72 = 0.$$

Here $b = -36$, $c = 72$, $d = -37$, and $e = 72$, and substituting in the relation (7′) we have

$$c(be - cd) - e^2 = 72^2(-36 + 37) - 72^2 = 0.$$

The relation (7′) is therefore satisfied, and the proposed equation has consequently two roots of the form $\pm\sqrt{-\beta}$. The values of these two roots are by (8′) $\pm\sqrt{-\frac{e}{c}} = \pm\sqrt{-1}$; and if the given equation be depressed by dividing the first side by $x^2 + 1$, the resulting cubic will be

$$x^3 - 37x + 72 = 0.$$

This example is taken from Bourdon's Algebre, p. 582, 1837, and Professsor Young has given the analysis of it in his Mathematical Dissertations, p. 161, ex. 8.

Lastly, let $e = 0$ in the relation (7), and also in the formula (8); then we obtain the relation

$$c(ab - c) = a^2 d \,\ldots\ldots\ldots (9), \qquad \text{and } \beta = \frac{ad}{ab - c} = \frac{c}{a} \,\ldots\ldots\ldots\ldots\ldots\ldots (10);$$

hence if $x^4 + ax^3 + bx^2 + cx + d = 0$ be an equation of the fourth degree, and if the values of the coefficients a, b, c, d, satisfy the relation (9), then the equation will have two roots equal to

$$\pm\sqrt{\left\{-\frac{ad}{ab - c}\right\}} \text{ or } \pm\sqrt{-\frac{c}{a}}.$$

Making $d = 0$ in the formula (9), gives the relation $ab - c = 0$, and hence if $x^3 + ax^2 + bx + c = 0$ be an equation of the third degree, and the values of a, b, c be such as to satisfy the relation $ab - c = 0$, then the equation has two roots of the form $\pm\sqrt{-\beta}$, and their values are $\pm\sqrt{-\frac{c}{a}}$, or $\pm\sqrt{-b}$, since $ab - c = 0$.

NOTE C.

In the last number of *The Mathematician* (No. 4, Vol. III, November, 1848), I gave a method of resolving a complete cubic equation without taking away its second term. The following investigation is analogous to that printed in the Mathematician, the only variation being a change of sign in the resulting formula for the value of the unknown quantity.

Let $x^3 + ax^2 + bx + c = 0$ be the given cubic equation, and let us assume

$$x^3 + ax^2 + bx + c = \frac{\lambda^3(x + y)^3 + (x + z)^3}{\lambda^3 + 1} \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Expanding the second member of (1); writing a' and b' for $\frac{a}{3}$ and $\frac{b}{3}$ respectively; and equating the coefficients of the same powers of x in both members, we get

$$\frac{\lambda^3 y + z}{\lambda^3 + 1} = a', \quad \frac{\lambda^3 y^2 + z^2}{\lambda^3 + 1} = b', \quad \frac{\lambda^3 y^3 + z^3}{\lambda^3 + 1} = c \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From these we obtain

$$\lambda^3 = -\frac{z - a'}{y - a'} = -\frac{z^2 - b'}{y^2 - b'} = -\frac{z^3 - c}{y^3 - c} \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Equating the first and second values of λ^3, and also the first and third, gives

$$yz + b' = a'(y + z) \ldots\ldots (4),$$

$$yz(y + z) + c = a'(y^2 + yz + z^2) \ldots\ldots (5).$$

Multiply (4) by $y + z$, and from the product subtract (5); then we have

$$a'yz + c = b'(y + z) \ldots\ldots (6).$$

From (4) and (6) we get

$$y + z = \frac{a'b' - c}{a'^2 - b'}, \text{ and } yz = \frac{b'^2 - a'c}{a'^2 - b'} \ldots\ldots (7).$$

The equations (7) will furnish the values of y and z, and from (3) we get

$$\lambda^3 = -\frac{z - a'}{y - a'}, \text{ or } \lambda^3(y - a') + z - a' = 0 \ldots\ldots (8).$$

Also from (1) we have $\lambda^3(x + y)^3 + (x + z)^3 = 0$, or $\lambda^3(x + y)^3 = -(x + z)^3$; hence

$$\lambda(x + y) = -(x + z) \ldots\ldots (9).$$

Eliminate z from the equations (8) and (9), and divide both members of the resulting equation by $\lambda + 1$; then we have

$$x = \lambda(\lambda - 1)(y - a') - a' \ldots\ldots (10).$$

If the form of the equation be $x^3 + bx + c = 0$, the modified equations are

$$y + z = \frac{3c}{b}, \quad yz = -\frac{b}{3}, \quad \lambda = -\left(\frac{z}{y}\right)^{\frac{1}{3}}, \text{ and } x = \lambda(\lambda - 1)y \ldots\ldots (11).$$

EXAMPLE.

Find the value of x in the cubic equation $x^3 + 12x - 30 = 0$.

Here $a = 0$, $b = 12$, $c = -30$, consequently we have $y + z = -\frac{15}{2}$ and $yz = -4$; hence

$$y = \frac{1}{2}, \quad z = -8, \quad \lambda = -\left(\frac{z}{y}\right)^{\frac{1}{3}} = 2\sqrt[3]{2}, \text{ and } x = \lambda(\lambda - 1)y = \sqrt[3]{2}(2\sqrt[3]{2} - 1) = 2\sqrt[3]{4} - \sqrt[3]{2}.$$

ERRATA.

Page 7, line 4 from bottom, *for* $r'(a + \beta)$ *read* $r'(a^2 + \beta)$.

— 8, — 3 from top, *for* equation (3) *read* equation (2).

— —, — 13 from bottom, *for* p. 4 *read* p. 6.

— 12, — 4 from top, *for* —434·67.... *read* —4·3467 ....

— 20, — 16 from top, *for* $f_6 a = 0$ *read* $f_4 a = 0$.

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